Chebyshev's inequality.

In this section we are aiming to give bounds on the prime counting function $\pi(x)$. We start with the more amenable

$$\psi(x) = \sum_{m \leq x} \Lambda(m)\,.$$

Lemma 2.12

$$\sum_{m \leq x} \Lambda(m) \left[\frac{x}{m}\right] = x \log x - x + O(\log x)\,.$$

Proof Evaluate the sum $\sum_{n \leq x} \log n$ in *two different ways.*

First, from (2) above $\log n = \sum_{m|n} \Lambda(m)$, so

$$\sum_{n \leq x} \log n = \sum_{n \leq x} \sum_{m|n} \Lambda(m) = \sum_{m \leq x} \Lambda(m) \sum_{\substack{n \leq x \\ m|n}} 1.$$

Here we have interchanged summations. We have not 'thrown away' any of the restrictions on m and n, instead we have *reinterpreted* them. For instance, the inner sum has gone from one over m, the *divisors* of n, to one over n, the *multiples* of m.

In the final inner sum, the condition $m|n$ means that n can be written as sm for some $s \in \mathbb{Z}$. Thus

$$\sum_{\substack{n \leq x \\ m|n}} 1 = \sum_{sm \leq x} 1 = \sum_{s \leq x/m} 1 = \left[\frac{x}{m}\right].$$

Hence

$$\sum_{n \leq x} \log n = \sum_{m \leq x} \Lambda(m) \left[\frac{x}{m}\right].$$

Alternatively, by Lemma 2.11 we have

$$\sum_{n \leq x} \log n = x \log x - x + O(\log x)\,.$$

Comparing these last two results gives the theorem. ■

We can now give an asymptotic result on a 'weighted form' of $\psi(x)$. The weight used is

$$w(u) = [u] - 2\left[\frac{u}{2}\right],$$

for $u \in \mathbb{R}$. Given $u \in \mathbb{R}$, let $m = [u]$, so $m \le u < m+1$. There are two cases: m even or odd.

If $m = 2n$, i.e. even then $2n \le u < 2n+1$, so $n \le u/2 \le n + 1/2$. Hence

$$\left[\frac{u}{2}\right] = n = \frac{m}{2},$$

which with $[u] = m$ gives

$$w(u) = [u] - 2\left[\frac{u}{2}\right] = m - 2\frac{m}{2} = 0.$$

If $m = 2n+1$, i.e. odd, then $2n+1 \le u < 2n+2$, so $n+1/2 \le u/2 < n+1$. Hence

$$\left[\frac{u}{2}\right] = n = \frac{m-1}{2},$$

which with $[u] = m$ gives

$$w(u) = [u] - 2\left[\frac{u}{2}\right] = m - 2\frac{(m-1)}{2} = 1.$$

Thus

$$w(u) = \begin{cases} 1 & \text{if } m \le x < m+1 \text{ for } \textit{odd } m \in \mathbb{Z} \\ 0 & \text{if } m \le x < m+1 \text{ for } \textit{even } r \in \mathbb{Z}, \end{cases}$$

a square-tooth function, period 2. In particular $0 \le w(u) \le 1$ for all $u \in \mathbb{R}$.

Lemma 2.13 *For* $x > 1$,

$$\sum_{m \le x} \Lambda(m)\, w\left(\frac{x}{m}\right) = x \log 2 + O(\log x)\,.$$

Proof By definition of the weight function,

$$\sum_{m \le x} \Lambda(m)\, w\left(\frac{x}{m}\right) = \sum_{m \le x} \Lambda(m) \left[\frac{x}{m}\right] - 2 \sum_{m \le x} \Lambda(m) \left[\frac{x}{2m}\right].$$

In the second sum consider the terms with $x/2 < m \le x$. Rearranging these inequalities we get

$$\frac{1}{2} \le \frac{x}{2m} < 1 \quad \text{and so} \quad \left[\frac{x}{2m}\right] = 0.$$

Thus these m can be discarded with no error, leaving

$$\begin{aligned}\sum_{m\leq x}\Lambda(m)\,w\Big(\frac{x}{m}\Big) &= \sum_{m\leq x}\Lambda(m)\left[\frac{x}{m}\right]-2\sum_{m\leq x/2}\Lambda(m)\left[\frac{x/2}{m}\right]\\ &= (x\log x-x+O(\log x))-2\left(\frac{x}{2}\log\frac{x}{2}-\frac{x}{2}+O(\log x)\right)\end{aligned}$$

by Lemma 2.12, applied twice, once with x and then with $x/2$. Hence

$$\sum_{m\leq x}\Lambda(m)\,w\Big(\frac{x}{m}\Big)=x\log 2+O(\log x)\,.$$

■

We now wish to remove the weight function w from the last result, but we can only do so at the cost of replacing the asymptotic result by upper and lower bounds.

Theorem 2.14 *For all $x>1$,*

$$\sum_{m\leq x}\Lambda(m)\geq(\log 2)\,x+O(\log x) \tag{7}$$

and

$$\sum_{x/2\leq m\leq x}\Lambda(m)\leq(\log 2)\,x+O(\log x)\,. \tag{8}$$

Thus

$$\psi\,(x)\geq(\log 2)\,x+O(\log x)$$

and

$$\psi\,(x)-\psi\,(x/2)\leq(\log 2)\,x+O(\log x)\,.$$

Proof Using the upper bound $w\,(u)\leq 1$ for all u within Lemma 2.13 gives

$$x\log 2+O(\log x)=\sum_{m\leq x}\Lambda(m)\,w\Big(\frac{x}{m}\Big)\leq\sum_{m\leq x}\Lambda\,(m)\,,$$

the first result of the theorem.

For the second result, (8), use Lemma 2.13 again but discard the terms $m\leq x/2$ from the sum, so

$$x\log 2+O(\log x)=\sum_{m\leq x}\Lambda(m)\,w\Big(\frac{x}{m}\Big)\geq\sum_{x/2<m\leq x}\Lambda(m)\,w\Big(\frac{x}{m}\Big)\,. \tag{9}$$

Here we have used the fact that $w(u) \geq 0$ and so we have discarded *non-negative* terms obtaining a *lower* bound.

For the remaining terms with $x/2 < m \leq x$, which rearranges first to $1 \leq x/m < 2$ for which $[x/m] = 1$. It also rearranges to $1/2 \leq x/2m < 1$ for which $[x/2m] = 0$. Thus

$$w\left(\frac{x}{m}\right) = \left[\frac{x}{m}\right] - 2\left[\frac{x}{2m}\right] = 1 - 2 \times 0 = 1,$$

and hence

$$x \log 2 + O(\log x) \geq \sum_{x/2<m\leq x} \Lambda(m)\, w\left(\frac{x}{m}\right) = \sum_{x/2<m\leq x} \Lambda(m)\,.$$

■

Aside The proof above lacks motivation at (9), why 'throw away' the integers $\leq x/2$? Answer: because I know what is coming next. Alternatively, because $w(u)$ is a square-tooth function, period 2, then

$$\begin{aligned} w\left(\frac{x}{m}\right) &= \begin{cases} 1 & \text{if } r \leq \dfrac{x}{m} < r+1 \text{ for } \textit{odd } r \\ 0 & \text{if } r \leq \dfrac{x}{m} < r+1 \text{ for } \textit{even } r \end{cases} \\ &= \begin{cases} 1 & \text{if } \dfrac{x}{r+1} \leq m < \dfrac{x}{r} \text{ for } \textit{odd } r \\ 0 & \text{if } \dfrac{x}{r+1} \leq m < \dfrac{x}{r} \text{ for } \textit{even } r \end{cases} . \end{aligned}$$

Thus

$$\begin{aligned} \sum_{m\leq x} \Lambda(m)\, w\left(\frac{x}{m}\right) &= \sum_{r \text{ odd}} \sum_{\frac{x}{r+1}\leq m<\frac{x}{r}} \Lambda(m) \\ &= \sum_{\frac{x}{2}\leq m\leq x} \Lambda(m) + \sum_{\frac{x}{4}\leq m<\frac{x}{3}} \Lambda(m) + \sum_{\frac{x}{6}\leq m<\frac{x}{5}} \Lambda(m) + \dots . \end{aligned}$$

There is no way to fill in the gaps $x/3 \leq m < x/2$, $x/5 \leq m < x/4$, etc. on the right hand side, so we look instead for upper and lower bounds on the left hand sum. For the upper bound we fill in all the gaps getting a complete sum of $\Lambda(m)$ over $m \leq x$. For the lower bound we 'throw away' all sums other than the first, over $x/2 \leq m \leq x$.

End of Aside

We now estimate from above the sum of $\Lambda(m)$ over **all** integers $m \leq x$, not just for $x/2 < m \leq x$. This is done at a cost of doubling the upper bound of $(\log 2)\, x$.

Corollary 2.15 *For all* $x > 1$,

$$\sum_{m \leq x} \Lambda(m) \leq (2\log 2)\, x + O\left(\log^2 x\right).$$

Combined with (7) and we have one form of **Chebyshev's inequality** (or sometimes **Čebyšev**) , namely

$$(\log 2)\, x + O(\log x) \leq \psi(x) \leq (2\log 2)\, x + O\big(\log^2 x\big). \tag{10}$$

Proof We split the sum over $m \leq x$ into a union of subintervals

$$\left[\frac{x}{2^{j+1}}, \frac{x}{2^j}\right],$$

for $j \geq 0$. If $x/2^j < 1$ this interval contains no integers so we can restrict $j \leq J$ where J satisfies

$$\frac{x}{2^{J+1}} < 1 \leq \frac{x}{2^J}.$$

Thus J is of size $O(\log x)$. Then we apply (8) in the midst of

$$\begin{aligned} \sum_{m \leq x} \Lambda(m) &= \sum_{j=0}^{J} \sum_{\frac{x}{2^{j+1}} < m \leq \frac{x}{2^j}} \Lambda(m) \\ &\leq \sum_{j=0}^{J} \Big((\log 2)\, \frac{x}{2^j} + O(\log x)\Big) \\ &= (\log 2)\, x \sum_{j=0}^{J} \frac{1}{2^j} + O(J \log x). \end{aligned}$$

The error term $J \log x = O\big(\log^2 x\big)$ while, for the main term, we complete the sum to infinity and sum the geometric series to gain the additional factor of 2. ■

Aside You might think that in the argument above we should have said

$$\sum_{j=0}^{J} \sum_{\frac{x}{2^{j+1}} < m \leq \frac{x}{2^j}} \Lambda(m) \leq \sum_{j=0}^{J} \left((\log 2)\, \frac{x}{2^j} + O\Big(\log\left(\frac{x}{2^j}\right)\Big)\right),$$

but this would have given no advantage, so we note that in the error term $\log(x/2^j) \leq \log x$ and continue as in the proof.

End of Aside

Note an interval of the form $[y, 2y]$ for any $y \in \mathbb{R}$ is called a *dyadic interval*. It is a common method in Number Theory to split an interval into a union of dyadic intervals.

The above result (10) was true for all $x > 1$ A sometimes more usable form is

Corollary 2.16 ***Chebyshev's inequality*** *Let $\varepsilon > 0$ be given. Then*

$$(\log 2 - \varepsilon)\, x < \psi(x) < (2\log 2 + \varepsilon)\, x \tag{11}$$

for $x > x_0(\varepsilon)$, i.e. for all sufficiently large x.

Proof The result $\psi(x) \geq (\log 2)\, x + O(\log x)$ above means that $\psi(x) \geq (\log 2)\, x + \mathcal{E}(x)$ for some function $\mathcal{E}$ satisfying $|\mathcal{E}(x)| < C \log x$ for some $C > 0$. Yet we know that logarithms grow slower than any power of x, so $C \log x < \varepsilon x$ for all $x > x_1(\varepsilon)$, i.e. x sufficiently large. Thus for such x we have

$$\mathcal{E}(x) > -C \log x > -\varepsilon x$$

in which case $\psi(x) \geq (\log 2)\, x - \varepsilon x$.

The upper bound in (11) follows from $\psi(x) \leq (\log 2)\, x + O\big(\log^2 x\big)$ in the same way, though perhaps with a different $x_2(\varepsilon)$. Choose $x_0(\varepsilon) = \max(x_1(\varepsilon), x_2(\varepsilon))$. ■

After all this work though, we will use Chebyshev's result below in the weak form $\psi(x) = O(x)$ for all $x > 1$ which follows from (10).

Relations between $\psi(x)$, $\pi(x)$ and $\theta(x)$; further Chebyshev inequalities

We could start with the simple observation that

$$\theta(x) = \sum_{p \le x} \log p \le \sum_{p^r \le x} \log p = \psi(x)\,. \qquad (12)$$

We can, though, prove an asymptotic result.

Lemma 2.17 *For $x \ge 2$,*

$$\psi(x) = \theta(x) + O\left(x^{1/2}\right), \qquad (13)$$

Proof From the definition of $\Lambda(n)$ as $\log p$ if $n = p^r$, 0 otherwise,

$$\psi(x) = \sum_{p \le x} \sum_{\substack{r \ge 1 \\ p^r \le x}} \log p = \sum_{r \ge 1} \sum_{p^r \le x} \log p$$

on interchanging the summations

$$= \sum_{r \ge 1} \sum_{p \le x^{1/r}} \log p = \sum_{r \ge 1} \theta\left(x^{1/r}\right).$$

This is, in fact, a finite sum since $\theta\left(x^{1/r}\right) = 0$ if $x^{1/r} < 2$, i.e. $r > \log x/(\log 2)$. Hence

$$\theta(x) < \psi(x) = \sum_{r \ge 1} \theta\left(x^{1/r}\right) = \theta(x) + \theta\left(x^{1/2}\right) + \sum_{r \ge 3} \theta\left(x^{1/r}\right). \qquad (14)$$

Thus

$$\begin{aligned} |\psi(x) - \theta(x)| &\le \theta\left(x^{1/2}\right) + \sum_{r \ge 3} \theta\left(x^{1/r}\right) \\ &\le \psi\left(x^{1/2}\right) + \sum_{r \ge 3} \psi\left(x^{1/r}\right) \qquad \text{by (12)} \\ &\ll x^{1/2} + \sum_{3 \le r \le \log x/\log 2} x^{1/r}, \end{aligned}$$

using Chebyshev's inequality in the form $\psi\left(x^{1/r}\right) \ll x^{1/r}$. We take the largest term out of this sum to get

$$\ll x^{1/2} + x^{1/3} \sum_{3 \le r \le \log x/\log 2} 1 \ll x^{1/2} + x^{1/3} \log x \ll x^{1/2}.$$

■

Check what would have happened if we had not taken the $r = 2$ term aside in (14).

We can then deduce another form of

Lemma 2.18 ***Chebyshev's inequality*** *For all* $\varepsilon > 0$

$$(\log 2 - \varepsilon)\, x < \theta(x) < (2\log 2 + \varepsilon)\, x$$

for all $x > x_3(\varepsilon)$.

Proof Not given, see Appendix ■

From this it is straightforward to prove

Corollary 2.19 *Given* $c > 2$ *the interval* $[x, cx]$ *contains a prime for all* x *sufficiently large, depending on* c.

Proof Subtracting the upper bound $\theta(x) < (2\log 2 + \varepsilon)\, x$ from the lower bound $\theta(cx) \geq (\log 2 - \varepsilon)\, cx$ gives

$$\begin{aligned} \theta(cx) - \theta(x) &\geq (\log 2 - \varepsilon)\, cx - (2\log 2 + \varepsilon)\, x \\ &= ((c-2)\log 2 - (1+c)\,\varepsilon)\, x. \end{aligned}$$

So choose ε to satisfy $(c-2)\log 2 - (1+c)\,\varepsilon > 0$. For example,

$$\varepsilon = \frac{(c-2)\log 2}{2\,(1+c)} > 0$$

would suffice. For then $\theta(cx) - \theta(x) > 0$, which means there exists a prime between x and cx. ■

Aside *Bertrand's Postulate* states that there exists a prime between x and $2x$ for all sufficiently large x. To prove this we would need a stronger form of Chebyshev's result. Chebyshev himself proved Corollary 2.16 with $\log 2 \approx 0.693471...$ replaced by $\kappa = \log\left(2^{1/2}3^{1/3}5^{1/5}30^{-1/30}\right) \approx 0.921292...$ and $2\log 2 \approx 1.386294.....$ replaced by $6\kappa/5 \approx 1.105550...$ respectively. These values would give the Corollary for any $c > 6/5$ and it would then include Bertrand's Postulate.

End of Aside

Having gone from $\psi(x)$ to $\theta(x)$ in Lemma 2.17 we now wish to go between the weighted sum $\theta(x)$ and the unweighted $\pi(x)$, i.e. between $\sum_{p\leq x} \log p$ and $\sum_{p\leq x} 1$.

This is achieved by Partial Summation.

Theorem 2.20 *For $x \geq 2$,*

$$\begin{aligned}\pi(x) &= \frac{\theta(x)}{\log x} + \int_2^x \theta(t)\,\frac{dt}{t\log^2 t} \\ &= \frac{\theta(x)}{\log x} + O\left(\frac{x}{\log^2 x}\right). \qquad (15)\end{aligned}$$

Proof Start with partial summation, so

$$\begin{aligned}\pi(x) &= \sum_{p\leq x} 1 = \sum_{p\leq x} \frac{\log p}{\log p} \\ &= \sum_{p\leq x} \log p \left(\frac{1}{\log x} - \left(\frac{1}{\log x} - \frac{1}{\log p}\right)\right) \\ &= \frac{1}{\log x}\sum_{p\leq x} \log p + \sum_{p\leq x} \log p \int_p^x \frac{dt}{t\log^2 t} \\ &= \frac{\theta(x)}{\log x} + \int_2^x \theta(t)\,\frac{dt}{t\log^2 t}.\end{aligned}$$

For the estimation of the integral apply (13) along with Chebyshev's upper bound to deduce $\theta(t) \ll t$. Then *splitting the integral at* $\sqrt{x}$ we get

$$\int_2^x \theta(t)\,\frac{dt}{t\log^2 t} \ll \int_2^x \frac{dt}{\log^2 t} = \int_2^{\sqrt{x}} \frac{dt}{\log^2 t} + \int_{\sqrt{x}}^x \frac{dt}{\log^2 t}.$$

In the first integral $t \geq 2$ so $\log t \geq \log 2$ and thus

$$\int_2^{\sqrt{x}} \frac{dt}{\log^2 t} \leq \frac{\sqrt{x}-2}{\log^2 2} = O\left(\sqrt{x}\right).$$

In the second integral $t \geq \sqrt{x}$ so $\log t \geq (1/2)\log x$ and thus

$$\int_{\sqrt{x}}^x \frac{dt}{\log^2 t} \leq \frac{4}{\log^2 x}\left(x - \sqrt{x}\right) = O\left(\frac{x}{\log^2 x}\right).$$

Here both integrals were estimated by

$$\int_a^b f(t)\,dt \leq \operatorname*{lub}_{[a,b]} f(t) \times (b-a).$$

Combining,

$$\int_2^x \theta(t)\,\frac{dt}{t\log^2 t} = O\big(\sqrt{x}\big) + O\left(\frac{x}{\log^2 x}\right) = O\left(\frac{x}{\log^2 x}\right).$$

■

Please Note This method of bounding an integral by splitting it at $\sqrt{x}$ is important and **should be remembered**. It works when the integrand changes a lot on the short interval $[1, \sqrt{x}]$, but changes little over the longer $[\sqrt{x}, x]$. This is a property of the logarithm, since $\log\sqrt{x} = (\log x)/2$.

We could have split the integral at x^α for **any** $0 < \alpha < 1$. We choose $\alpha = 1/2$ simply as the "simplest" number less than 1.

Now we come to the final version of

Corollary 2.21 ***Chebyshev's inequality*** *For all* $\varepsilon > 0$

$$(\log 2 - \varepsilon)\,\frac{x}{\log x} < \pi(x) < (2\log 2 + \varepsilon)\,\frac{x}{\log x}$$

for all $x > x_4(\varepsilon)$.

Proof Not given, see Appendix ■